# Bordered Hessian Matrix

1. Recall:

   A Hessian Matrix is a matrix of all second-order partial derivatives and it is used to determine if a critical point is a local min, local max or saddle point.

2. Def:

   A Bordered Hessian Matrix, denoted by $\overline{H}$, is used to determine if a constrained crit point is a local min or local max.

3. Thm:

   Let $f: U \subset \mathbb{R}^2 \to \mathbb{R}$ and $g: U \subset \mathbb{R}^2 \to \mathbb{R}$ be smooth (at least $C^2$) functions. Let $\vec{v_0} \in U$, $g(\vec{v_0}) = c$ and $S$ be the level curve for $g$ with value $c$. Assume that $\nabla g(\vec{v_0}) \neq \vec{0}$ and that there is a real number $\lambda$ s.t. $\nabla f(\vec{v_0}) = \lambda \nabla g(\vec{v_0})$. Let $L = f - \lambda g$. Then, the Bordered Hessian Matrix determinant, denoted by $|\overline{H}|$, is given by:

$$|\overline{H}| = \left\| \begin{matrix} 0 & -\frac{\partial g}{\partial x} & -\frac{\partial g}{\partial y} & \cdots \\ -\frac{\partial g}{\partial x} & \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} & \cdots \\ -\frac{\partial g}{\partial y} & \frac{\partial^2 L}{\partial y \partial x} & \frac{\partial^2 L}{\partial y^2} & \\ \vdots & \vdots & \vdots & \end{matrix} \right\| (\vec{v_0})$$

If $|\bar{H}|>0$, then $\vec{V_0}$ is a local max.
If $|\bar{H}|<0$, then $\vec{V_0}$ is a local min.
If $|\bar{H}|=0$, then the test is inconclusive.